THE

MATHEMATICAL GAZETTE

A TERMINAL JOURNAL FOR STUDENTS AND TEACHERS.

Edited by E. M. LANGLEY, M.A.

PUBLISHED BY MACMILLAN AND CO., LONDON AND NEW YORK.

No. 5.] MAY 1895. [PRICE ONE SHILLING NET.

THE SYLLABUS OF GEOMETRICAL CONICS.

THE Sixth General Meeting of the Association for the Improvement of Geometrical Teaching appointed a Committee for Geometrical Conics, and one for "Higher Plane Geometry, including such subjects as Transversals, Projection, Anharmonic Ratio, etc."

Dr. Hirst in his Presidential Address has said of the latter subjects, "Until these notions become more familiar ones, I, for my part, believe that Geometrical Conics will always remain in its present unsatisfactory condition. . . . It will be, of course, a question for this Association to decide whether, pending the introduction of the more thorough treatment based on the notions to which I have alluded, some improvement may not be introduced into the subject of Geometrical Conics, as at present understood, with a view of enabling examiners at all events to examine with greater facility and purpose. At present the definitions are so multiform and the sequence of propositions so varied in different text-books, that it is found to be an exceedingly difficult task to examine satisfactorily in the subject at all."

Before considering how to give effect to the principle of modernising the subject in a practical way, let me deprecate any narrowness of view with respect to elementary geometry. It is not merely a first step to a higher geometry. It is also, beside being an instrument of mental culture, an introduction to applied mathematics and the solution of physical problems; and the clear insight into the laws and processes of nature enjoyed by Kepler and Newton, and in our own days by such men as Adams and Maxwell, is inseparable from their previous mastery of the geometry of Euclid, Archimedes, and Apollonius.

That I should be proposing just now to discuss the Association's Syllabus of Geometrical Conics is owing partly to the fact that I have, as I think, arrived at something like finality in my own view of the way in which the subject should be approached. I may add that the Committee, before reporting, did me the honour to ask me to join it as a co-opted member, so that I have been, as I may say, invited to express an opinion on the subject. It seems to me that I can do this rather better now than I could have done then.

To go back to the beginning of my work at geometrical conics. The first thing that I wrote, namely in 1862, was an article in the *Oxford, Cambridge, and Dublin Messenger of Mathematics* (vol. i. p. 250), which commenced with the remark that the results of orthogonal projection can be proved in one plane, namely, by what the A. I. G. T. Syllabus aptly terms *Reduction of Ordinates*. This is, I am sure, a safer construction for the beginner, but orthogonal projection must of course come in somewhere. Let me read a paragraph from the Thirty-Sixth Annual Report of the Cambridge Local Examinations and Lectures Syndicate, dated 10th March 1894 (see the *Cambridge University Reporter* for 1893-94, p. 596).

"*Conic Sections.* Seniors:—A few very good papers were sent up in this subject, but most of the candidates were only able to write out one or two of the pieces of book-work. Many candidates attempted to prove one of the geometrical theorems by projections; most of these proofs were, however, valueless, for all that was done was to prove at unnecessary length a well-known property of the circle, no appeal being made to any of the fundamental principles of projection."

The same article in the *Oxford, Cambridge, and Dublin Messenger* contains the proposition, which I had not seen stated anywhere, that, at any point of a central conic,—

The projection of the normal, terminated by the minor axis, upon either focal distance, is equal to half the major axis.*

Let the circle through the foci S, H, and a point P on the curve cut the minor axis in g and t.

The lines Pg, Pt are the bisectors of the angle SPH.

Let Pg be the internal bisector, and let the conic be an ellipse. Then Pg is the normal at P.

Draw gE, gK normal † to SP, HP respectively.

Then evidently $gE = gK$.

* Note that the square of the true conjugate axis of a hyperbola is negative, and therefore less than the square of the transverse axis.

† The expression "normal" is a convenient substitute for "perpendicular," and there is precedent for its use in this sense.

Also $$gS = gH.$$

Hence, in the right-angled triangles gES, gKH,

$$SE = HK,$$

or $$SP - PE = PK - HP.$$

Hence, PK being equal to PE,

$$PE \text{ (or } PK) = \tfrac{1}{2}(SP + HP).$$

Next, let the conic be a hyperbola.

In this case Pt is the normal at P.

Draw tE', tK' normal to SP, HP; then it may be shown similarly that

$$PE' \text{ (or } PK') = \tfrac{1}{2}(SP \sim HP).$$

In either case the projection of the normal upon SP or HP is equal to CA, or half the major axis.

Corollaries.

1. Given that EK is the diameter parallel to the tangent at P, it follows by right-angled triangles, if Pg, EK cross at F, that

$$PF \,.\, Pg = PE^2 = CA^2.$$

2. Given, further, that the projection PL of the normal terminated by the major axis upon SP is equal to CB^2/CA, then similarly

$$PF \,.\, PG = PE \,.\, PL = CB^2.$$

3. By similar triangles, if CD be the radius conjugate to CP,

$$PG \,.\, Pg = SP \,.\, HP = CD^2.$$

4. If PN be the ordinate of P, and e denote the eccentricity, then, because CG, SE and CS, PE subtend equal angles at g,

$$CG : SE = CS : PE,$$

or $$CG = e \,.\, SE = e^2 \,.\, CN.$$

In 1863 was published my *Geometrical Conics, including Anharmonic Ratio and Projection*, in which a chapter upon "Conics" in general came before the chapters on the three separate curves. The chapter on "Conics" began with the tangent, and gave the well-known construction for tangents by Professor Adams.

Looking back at an article on "The Principles of Geometrical Conics," in the *O. C. D. Messenger*, vol. v. (1871), I find this note on what I called Adams' theorem: "I have varied the proof. That given by the discoverer being indirect."

Professor Adams, as I am reminded by this note, took Euclid's definition of a tangent, and obtained some of his results in a very indirect way.*

From any point T on a certain line through P on a conic he drew (Fig. 2) perpendiculars TM, TN to SP and the directrix, and proved that

$$SM = e \,.\, TN.$$

Hence, ST being greater than TN, and T therefore lying without the conic, except when T is at P, the line TP is the tangent at P.

To draw the tangents from a given point T, draw the circle with centre S and radius equal to $e \,.\, TN$; draw tangents TM, TM′ to the circle, and SR, SR′ parallel to them to meet the directrix. This may be shortened a little by calling the circle the Adams circle.

Then TR, TR′ are the required tangents; and TP, TP′ subtend equal (or supplementary) angles at S, because in the circle the angles TSM, TSM′ are equal.

This is a variation upon the construction of Boscovich, now well known to members of the Association through Mr. Langley's articles upon the "Eccentric Circle," as I have proposed to call Boscovich's circle.

The general chapter on conics in my *Geometrical Conics* of 1863 contained a proof of the property of diameters, of which the following was the original form.

Let PQ be a chord given in direction, O its middle point, R its intersection with the directrix (Fig. 3).

Draw SY normal to the chord, and let it meet the directrix in D.

Then $$SP : PR = SQ : QR = \text{a constant ratio } (c).$$

Hence $$c^2 (PR^2 \sim QR^2) = SP^2 \sim SQ^2 = PY^2 \sim QY^2.$$

This reduces to $$OY/OR = c^2;$$

whence it follows that the locus of O is a straight line through D.

This led up to a new proof that

$$CV \,.\, CT = CP^2,$$

including $PV = PT$ in the parabola.

I will now submit some observations upon the Syllabus, beginning with its traditional treatment of the diameters of a hyperbola which do not meet the curve.

The Conjugate Hyperbola.

Mr. Charles Smith, Master of Sidney Sussex College, writes in the preface to his recently published *Geometrical Conics* (1894): "I have discarded the usual method of treating a hyperbola as if it were *two* conics, namely, the curve itself and the conjugate hyperbola. This will, I hope, meet with the approval of teachers."

But in this there is no novelty. In the preface to edition 1, part 1 of the *Elementary Geometry of Conics*, which was dated April 1872, I wrote: "No allusion has yet been made to the Conjugate Hyperbola, which may be viewed as a contrivance for giving a false definiteness to the student's conceptions, and perpetuating his illusion that the Hyperbola is a discontinuous curve."

Eventually the conjugate hyperbola was mentioned in an Appendix. In the present edition it is defined only in a footnote to Problem 259, after mention as below in the last paragraph of the text (p. 108): "The diameters of a hyperbola which do not meet it in real points are sometimes said to be terminated by the conjugate hyperbola (Prob. 259). But this brings in an unnecessary curve and is apt to mislead the student."

A little before the year 1881 I found—and wrote in the prolegomena to the *Ancient and Modern Geometry of Conics*—that Boscovich condemned the conjugate hyperbola.

* Later writers have taken them (with the proof that $SG = e \,.\, SP$) in the form in which they found them in my *Geometrical Conics* (1863). But see the preface to Mr. Richardson's *Geometrical Conics*.

This unnecessary curve is commonly brought in to prove that, with the usual lettering,

$$CP^2 \sim CD^2 = CA^2 \sim CB^2;$$

that is to say, $CP^2 \sim PT^2 = CA^2 - CB^2$,

if the tangent at P meets an asymptote in T.

Leaving out the curve altogether, we have to prove that if CT, Ct be lines given in position, and if the rectangle CT, Ct be constant, then

$$CP^2 \sim (\tfrac{1}{2}Tt)^2 = \text{a constant}.$$

Bisect CT, Tt in O, P, and draw Py normal to CT.

Then
$$\begin{aligned} CP^2 \sim PT^2 &= Cy^2 \sim Ty^2 \\ &= 4OC \,.\, Oy, \\ &= \text{a constant}, \end{aligned}$$

because (in the triangle OPy, whose sides are given in direction) Oy varies as OP, and therefore inversely as OC.*

Having proved in this way that $CP^2 \sim CD^2$ is constant, we can dispense with the preceding proposition in the Syllabus, and with the eight excellent four-branch figures given by Messrs. Cockshott and Walters in their chapter on the Hyperbola.

We may then show by the eccentric circle, as in the *Elementary Geometry of Conics* (p. 67 note), how the two branches of the hyperbola join at infinity, thus modernising the geometry of that curve, while we also gain greatly in simplicity.

The conjugate hyperbola is an antiquated stumbling-block in the way of sound teaching.

The Parabola.

The proposition $QV^2 = 4SP \,.\, PV$

is presumably meant to be proved by tangent-properties, and Messrs. Cockshott and Walters, who follow the Syllabus, do so prove it. But it can be proved very simply indeed without them, by steps which can be retraced with equal ease, as in a corollary in the *Elementary Geometry of Conics* (p. 35).

It is worth while to show the student of mechanics how to work back from "QV^2 varies as PV" to the focus and directrix of the parabola.

The Ellipse.

It does not appear from the Syllabus how

$$QV^2 : PV \,.\, P'V = CD^2 : CP^2$$

is to be proved,—whether by tangent-properties or without them,—if any of the preceding propositions may be used. There should be the same rule for the ellipse and the parabola. But the Syllabus does not appear to rest as a whole upon any principles.

The Hyperbola.

Mr. Cockshott, speaking of the Syllabus at the Ninth General Meeting of the Association, said that "The asymptotes were introduced into the hyperbola rather earlier than usual. They really got very little idea of the shape of the hyperbola without the asymptote" (Report for 1883, p. 21). About this there can be no doubt. Probably the hyperbola was discovered from the asymptotes as the locus "xy = a constant," as is argued in a Paper printed in the Association's Report for 1884.

The *New Treatment of the Hyperbola*, printed in the Report for 1890, and afterwards in the *Elementary Geometry of Conics*, brings in the asymptotes still earlier, and uses them to prove that

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2.$$

First it is shown that, if an asymptote meets the directrix in Y and the ordinate in O,

$$OY = SP;$$

and then, by applications of Euclid I. 47, that

$$ON^2 - PN^2 = SY^2 \text{ (or } CB^2),$$

whence we get $PN^2 + CB^2$ in terms of CN^2 by parallels.

This construction is perhaps the simplest possible. Thus far we want only one quadrant of the curve and one vertex; and the steps of the proof being reversible, it is easy to work back by it from "xy = a constant" to the focus and directrix.

In the Paper of 1884 it was said of the conjugate hyperbola that "It comes down to us from times when the doctrine of continuity in relation to infinity had not as yet been formulated, and in so far as it is regarded as useful or necessary it is a hindrance to sound teaching in the matter of infinity, and to a right conception of the genesis of the hyperbola."

Inventio Orbium.

Newton showed how to determine conics, which he regarded as planetary orbits, from given points upon them and other conditions. In pure geometry it is equally necessary to be able to show that a conic is determined, for example, by five points; and a Syllabus is incomplete without at least this primary problem.

Attention is called to this grave defect in the Syllabus of Geometrical Conics by a recent work on Modern Plane Geometry, in a footnote to the proof that "the locus of a point at which four fixed points subtend a constant cross ratio is a conic," thus: "There is, of course, an incompleteness in this proof if it may not be assumed that a conic can be described through five given points, and this proposition *is not proved in treatises on Geometrical Conics.*"

This is true of a book which follows the Syllabus, but not of all books upon the subject. The young student, however, who cannot be expected to know better than his text-book, is given to understand that an absolutely essential proposition is to be taken for granted, and that it passes the wit of man to prove it in a presentable way. Is this "the more thorough treatment" which modern geometers desiderate?

Professor Adams found by experience as an examiner

* Another form of the proof is as follows. Complete the parallelogram TCtK, and let x, y denote its sides. Then $CK^2 - Tt^2 = x^2 + y^2 + 2xy \cos C - (x^2 + y^2 - 2xy \cos C)$, which is constant when xy and C are given.

that a little knowledge of modern geometry, with its "circular points at infinity," is a fruitful source of error.

Recent Works.

As in 1871 (*O. C. D. Messenger*, vol. v. p. 143), I should take as the first principle in an elementary work, that propositions should be proved *as nearly from the definition and with as few assumptions as possible*. This carries with it that chord-properties should in general be proved without the help of tangent-properties.

A recent writer completely adopts my way of working out the latter principle in the case of the parabola, and follows the *Geometry of Conics* in other respects.

Of all possible improvements in detail in the Syllabus the disuse of the conjugate hyperbola seems the most important. It has been already mentioned that Mr. C. Smith discards it.

Messrs. Milne and Davis make no use of the conjugate hyperbola.

When this has been got rid of, the student can be introduced by means of the eccentric circle to the leading ideas of modern geometry.

C. Taylor.

MATHEMATICAL WORTHIES.

II. John Dee.

John Dee, commonly though improperly called Dr. Dee, whose life extended over more than eighty years (1527-1608), was a mathematician and astronomer for more than fifty years, and a magician and raiser of spirits in his old age. It is by these latter practices that he is generally best known, and full accounts of them are given in the *Dictionary of National Biography* and in the life of Sir Edward Kelley in the recently published book, *Twelve Bad Men*, edited by Mr. Seccombe.

Here we have to do with him as a mathematician. He was one of the original Fellows of Trinity College, Cambridge, being then but twenty, and seems to have incurred the reproach of practising magical arts through having devised some clever stage tricks for a representation of the Pax of Aristophanes. He went abroad and lectured on Euclid at Louvain and Rheims, being so popular at the latter place that those for whom there was no room struggled for places to look in at the windows.

The most noteworthy contribution that he made to mathematical science was his preface and notes to the splendid translation into English by Sir H. Billingsley of the fifteen books ascribed to Euclid. De Morgan has stated his belief that this translation was really Dee's work. The preface, an account of the state of mathematical science at that time, is worth reading from its historical interest. Dee disliked the word geometry, and would have preferred megethology in its place. His catalogue of applications of geometry is perspective, astronomy, music, cosmography, astrology, static, anthropography, trochilike, helicosophy, pneumatithmy, menadry, hypogeiody, hydragogy, horometry, zography, architecture, thaumaturgike, archimastry. It may be as well to explain that menadry treats of the mechanical powers, hypogeiody refers to the projection on the irregular surface of the earth of the course of a mining passage, and that zography is painting. In his account of astronomy we find it stated that the sun is $161\frac{7}{8}$ times greater than the earth, and the earth is $42\frac{7}{8}$ times greater than the moon; that, estimated in radii of the earth, the sun's greatest distance is 1179, the moon's greatest distance is $68\frac{1}{3}$ and least $52\frac{1}{4}$, while the distance of the starry sky is $20081\frac{1}{2}$. De Morgan in the companion to the *Almanack* 1837 and 1855 notices that there is no statement here of his adherence to the Copernican hypothesis, but he thinks that Recorde, Dee, and Digges almost certainly held that view, though unavowedly.

Queen Elizabeth found Dee very useful to her. He was a man of ability, and under the character of one interested in science and astrology could really engage, without being suspected, in secret political missions. Lilly distinctly says he received a regular salary, but it is more likely that he was economically paid with mere promises.

When the Pope was considering the subject of the reformation of the Calendar, Dee was engaged by the Queen to report upon the matter, and in Halliwell's collection of letters on science there is a memorial of Lord Burghley's concerning this report. So far as England was concerned, the feeling against Popery was too strong to allow the reformation to be made.

Dee acted in anticipation of modern times in the petitions he addressed to both Mary and Elizabeth for the preservation and recovery of ancient writings and monuments. He died in 1608, and was buried at Mortlake.

G. Heppel.

NOTES.

18. On the hyperbolic functions.

Hyperbolic functions may be used in problems relating to a single branch of the hyperbola, and they are specially useful in investigating the properties of a chord, since by their aid the hyperbolic formulæ become similar to those of the ellipse.

It may therefore be of service to give a list of cases where the use of these functions is preferable to the substitution $x = a \sec \phi$, $y = b \tan \phi$.

A point P on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be taken as $(a \cosh u, b \sinh u)$, where $u = 2$ area ACP$/ab$.

The tangent at (u) is $\frac{x}{a} \cosh u - \frac{y}{b} \sinh u = 1$.

The normal at (u) is $ax \operatorname{sech} u + by \operatorname{cosech} u = a^2 + b^2$.

The chord QQ′ through $(u \pm v)$ is $\frac{x}{a} \cosh u - \frac{y}{b} \sinh u = \cosh v$.

The mid-point V of QQ′ is ($a \cosh u \cosh v$, $b \sinh u \cosh v$).

T, the pole of QQ′, is ($a \cosh u \operatorname{sech} v$, $b \sinh u \operatorname{sech} v$).

The equation to CV is $\frac{x}{a \cosh u} = \frac{y}{b \sinh u}$: hence P, the point ($u$) on the curve, lies on this line, so also does T.

Hence, since CT, CP, CV are propl. to their abscissæ or ordinates, $CV \cdot CT = CP^2$.

The tangent at P is parallel to QQ′.

A point on the conjugate hyperbola may be taken as ($a \sinh u'$, $b \cosh u'$).

If CD is conjugate to CP, D is ($a \sinh u$, $b \cosh u$), $CV = CP \cdot \cosh v$, $QV = CD \cdot \sinh v$,

$$\therefore \frac{CV^2}{CP^2} - \frac{QV^2}{CD^2} = 1, \quad \text{or} \quad \frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

with CP, CD as axes.

With these axes a point Q on the curve is ($a' \cosh v$, $b' \sinh v$).

The tangent at v is $\frac{x}{a'} \cosh v - \frac{y}{b'} \sinh v = 1$.

The chord through $(u \pm v)$ is $\frac{x}{a'} \cosh u - \frac{y}{b'} \sinh u = \cosh v$.

W. GALLATLY.

19. On the "circle of Apollonius."

In the discussion which followed Mr. Milne's paper on "the Conics of Apollonius" at the last General Meeting of the A.I.G.T., a question was asked as to the historical propriety of the above name for the locus of a point whose distances from two given points are in a constant ratio. The following note from Montucla may be found interesting: "Comme nous avons dit sur cet ouvrage d'Apollonius (de locis planis) des choses capables de piquer la curiosité des Géomètres, nous en allons extraire quelques-unes des propositions les plus remarquables.

(i) Que si d'un point comme P partent plusieurs paires de lignes PA, PB; Pa, Pb, etc. faisant des angles constans APB, aPb, etc. et que la raison de celles de chaque paire ou leur rectangle soit invariable si les extrêmités A, a sont dans un lieu plan, les autres B, b etc. seront aussi dans un lieu plan, c'est-à-dire, dans une ligne droite ou circulaire.

(ii) Si l'on a tant de lignes données de position qu'on voudra et que d'un point P on leur tire des lignes sous des angles donnés et que la somme de deux soit à la troisième ou la somme de deux à celle des deux restantes, etc. en raison donnée, le point P sera dans un lieu plan, c'est-à-dire que tous les points d'où partiront des lignes sous ces conditions seront dans une ligne droite ou circulaire.

(iii) *Si de deux points donnés* A, B *sont tirées à un troisième* P *deux lignes* AP, BP *qui soient dans une raison d'inégalité, ce point* P *est dans une circonférence circulaire où tous les points* P, p, π, *etc. où se terminent des lignes dans cette raison, forment une circonférence de cercle.* Ce sera encore un cercle si le quarré de l'une augmenté ou diminué d'un quarré constant est au quarré de l'autre en raison donnée.

(iv) Si de tant de points qu'on voudra partent autant de lignes à un point P et que la somme des quarrés de ces lignes soit invariable, ce point P et tous ses semblables sont dans un cercle, dont on determinera le centre dans cette manière. Que A, B, C, D, E, F soient les points donnés, nous en supposons 6; ayant tiré la ligne NO qui les laisse tous d'un côté que AG, BH, etc. lui soient perpendiculaire, et que GT soit la 6^e partie des lignes GH, GI, etc. jusqu'à GM inclusivement (ou NT la 6^e partie des six lignes NG, NH, NI etc.) et que l'on tire la perpendiculaire PQ qui soit aussi la 6^e partie des six lignes AG, BH, etc., le point Q sera le centre du cercle; et son rayon QR sera tel que $6QR^2 + AQ^2 + BQ^2$, etc. soient égaux au quarré donné, ou au six quarrés AP^2, BP^2, etc.

De là il est facile de voir que le point Q est le centre de gravité des points A, B, C, etc., et si l'on supposoit que le double du quarré de AP, plus le triple de BP, plus le cinquième de CP fissent ensemble un quarré constant, le centre du cercle où seroient les points P seroit encore le centre de gravité des points A, B, C, mais en supposant le point A chargé d'un poids double, B d'un triple, etc. C'est M. Huygens qui a remarqué cette consequence de la détermination d'Apollonius." EDITOR.

SOLUTIONS OF "GAZETTE" QUESTIONS (1-17).

(*When not otherwise stated the solutions given are by* Mr. A. E. FIELD *and the* EDITOR.)

1. A, B, C *are three points on a circle whose centre is* O *such that arc* AB *is double of arc* BC. *If* AH, BK *are drawn perpendicular to* OB, OC, *show that* HK *is parallel to* AB. EDITOR.

Draw OE perpendicular to AB. Then $\angle BOE = \frac{1}{2} \angle AOB = \angle BOC$, and Δ^s OBE, OBK are congruent. Hence also so are Δ^s BHE, BHK.

$\therefore \angle BHK = \angle BHE = \angle OAE$ ($\because$ OHEA is cyclic)
$= \angle HBE$.

therefore HK is parallel to AB. FLAMINGO.

2. PHQ, POQ *are two triangles on opposite sides of the base* PQ *such that angles* QHP, HPQ = *angles* OQP, QPO *respectively. Any parallel to* OQ *cuts* PO, PQ *in* L, M. *Show that* QL, HM *intersect on a fixed circle.* EDITOR.

Let R be the point of intersection of QL and HM. Join PR. Then

PL : PM :: PO : PQ
:: PQ : PH, by similar triangles,
$\therefore$ PL : PQ :: PM : PH.

Therefore the triangles PQL and PMH have the angles at P equal and the sides about them proportional, and are therefore equiangular, so that angle PQL = angle PHM.

Therefore P, H, Q, R are concyclic.

Therefore R lies on the circle through P, Q, H.

FLAMINGO.

3. OP, OQ *are tangents to a circle;* PO *is produced to* F *so that* QF = PQ. *From a point* M *on* PQ, MK, ML *are drawn parallel to* OP, OQ *meeting* QF, OP *in* K, L. *Show that* PK, QL *intersect on the circle.* EDITOR.

Let PK, QL intersect at N.

By similar Δ^s, PF : FQ :: QP : PO.

But FQ : FK :: PQ : PM,

:: PO : PL,

$\therefore$ PF : FK :: QP : PL.

But $\angle PFK = \angle QPL$,

$\therefore \angle FPK = \angle PQL$,

and $\angle PLN$ is common to Δ^s PLN, LQP,

$\therefore \angle LNP = \angle OPQ$,

therefore N lies on the given circle.

4. *In an ellipse two radii* CL, CM *are drawn at right angles to each other, and* CQ *is perpendicular to* LM. *Show geometrically that the locus of* Q *is a circle.* EDITOR.

It is easy to show that $\frac{1}{CL^2} + \frac{1}{CM^2}$ is constant.

Now CQ . LM = CL . CM, for each product is twice the area of the triangle CLM.

$$\therefore CQ^2 (CL^2 + CM^2) = CL^2 . CM^2.$$

Hence CQ^2 is constant, being equal to the first expression.

Or thus by Flamingo :—Reciprocate with respect to the centre of the ellipse the known theorem that the locus of the intersection of tangents to the ellipse at right angles to one another is a circle.

5. *Given* $17^2 = 12^2 . 2 + 1$, *so that* $\sqrt{2} = 17 \div 12$ *approximately, deduce other closer approximations of the same nature without using continued fractions.* EDITOR.

$$17^2 - 1 = 12^2 . 2.$$

Multiply by $4 . 17^2$, and add unity,

$$\therefore 4 . 17^2(17^2 - 1) + 1 = 4 . 17^2 . 12^2 . 2 + 1$$

$$\therefore (2 . 17^2 - 1)^2 = (2 . 17 . 12)^2 . 2 + 1$$

$$\therefore 577^2 = 408^2 . 2 + 1,$$

so that $\sqrt{2} = 577 \div 408$, approximately.

The process may be repeated indefinitely.

6. *In a letter to the* "English Mechanic" *Lord Grimthorpe gives the following numbers for the construction of approximately regular pentagons: "If side* = 13 *then the diagonal* = 21 *and the radius of the circumcircle* = 11." *In the* "Illustrated Carpenter and Builder" *I find that the "mitre" for such a pentagon is* $13\frac{3}{4} \times 10$. *Show that there exists a simple geometrical connection between the two results.* EDITOR.

The "mitre" gives two adjacent sides of a right-angled triangle, whose greater acute angle is half one of the angles of a regular pentagon.

Let ABCDE be a regular pentagon, O the centre of the circumcircle, K the intersection of the tangents to this circle at C and D. Let BD and CE intersect at H; then OH passes through K. Join OD. Then it is known that BH = CD.

Hence, taking Lord Grimthorpe's numbers, BH = 13 and HD = 8. But CHDK is a rhombus, therefore DK = 8.

$$\therefore \tan \tfrac{1}{2}A = \tan OKD = \frac{OD}{KD}$$

$$= \frac{11}{8} = \frac{55}{40} = \frac{13\frac{3}{4}}{10}.$$

7. *The general formula for three integers which may be taken to represent the length of the sides of a right-angled triangle is well known to be* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$. *Show that such sets of lines occur simply in the geometry of the parabola.* EDITOR.

With the usual letters, let the ratio of SY : SA have any commensurable value $x : y$.

Then, if SA be divided into y^2 parts, SP will contain x^2 of these parts, for SP . SA = SY^2.

Also the measure of PG will be $2xy$, of AG $x^2 + y^2$, and of AN $x^2 - y^2$. Hence we are led to the theorem

$$AG^2 = AN^2 + PG^2,$$

which is easily demonstrated thus :—

$$AG^2 - AN^2 = YG^2 - YN^2 = YG^2 - YP^2 = PG^2.$$

8. *A series of positive numbers* $u_1, u_2, u_3 \ldots u_n$ *is constructed under the condition* $u_n = u_{n-1} + u_{n-2}$, u_1 *and* u_2 *being taken arbitrarily. Find the limiting value of the fraction* $u_n \div u_{n-1}$ *as n increases indefinitely.* EDITOR.

$$\frac{u_n}{u_{n-1}} = 1 + \frac{u_{n-2}}{u_{n-1}}$$

$$= 1 + \frac{1}{1 + \frac{u_{n-3}}{u_{n-2}}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{u_{n-4}}{u_{n-3}}}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \cdots + \frac{u_1}{u_2}}}.$$

Let x = required limit,

$$\therefore x = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}} \text{ to infinity}$$

$$= 1 + \frac{1}{x}$$

$$\therefore x^2 - x - 1 = 0,$$

and x is necessarily positive,

$$\therefore x = \frac{\sqrt{5} + 1}{2}.$$

Note the connection between this result and the quotation from Leslie's *Elements of Geometry*, on page 6, No. 1, of the *Gazette*.

9. PN *is the ordinate of an ellipse. Draw any straight line* CH *from* C *equal to the semi-major axis. Draw* AK *parallel to* HN *to meet* CH *in* K. *Produce* CN *to* T, *making* CT = CK. *Then* PT *is the tangent at* P. W. J. GREENSTREET.

Since KA is parallel to HN,

$\therefore$ CK : CH :: CA : CN,

$\therefore$ CT : CA :: CA : CN,

$\therefore$ PT is the tangent at P.

10. *Show that with the same constant and pole of inversion with which either of two given circles may be inverted into the other, their circle of similitude inverts into their radical axis.* H. D. ELLIS.

Let I and E be the internal and external centres of similitude of two circles, centres at A and B. Draw the radii AS, BT to the points of contact of an external common tangent EST, which meets the circle of similitude at P and the radical axis at Q.

Then, since the angle IPS is right, therefore AS, IP, BT are parallel. But, since BI : IA :: BE : EA, BIAE is a har-

monic range, therefore TPSE is a harmonic range. But Q is the mid point of ST,

$$\therefore EP \cdot EQ = ES \cdot ET,$$

which proves the theorem.

11. *In a triangle* ABC *it is given that* $AB - AC = \frac{1}{2}BC$. *If* D *be taken on* BC *such that* $BD = \frac{1}{4}BC$, *show that angle* ACB *is double of angle* ADC.

Prof. J. E. A. STEGGALL.

Obviously, $BD = s - c$, and therefore D is the point of contact of BC with its escribed circle. Join AD, and, of its two points of intersection with the inscribed circle, let H be the one nearer A.

Then, since A is the external centre of similitude of these two circles, the tangent at H to the inscribed circle is parallel to BC.

Take a point L on CB such that $CL = BD = \frac{1}{4}BC$; then, since $CL = s - c$, L is the point of contact of BC with the inscribed circle. Since the tangents at H and L are parallel, HL is a diameter of the inscribed circle, and its mid point I is therefore its centre. Join CI, and draw IK parallel to CB to meet AD at K. Then HD is bisected at K, and $KI = \frac{1}{2}DL = LC$. Also, since the circle on HD as diameter passes through L, KD = KL.

$$\therefore \text{angle } KDL = \text{angle } KLD$$
$$= \text{angle } ICL$$
$$= \tfrac{1}{2} \text{ angle } ACB.$$

12. *A straight line cuts a conic in* P, P′ *and the directrix in* Z. *From any point* K *on it is drawn a perpendicular* KI *to the directrix, and a parallel* KUU′ *to* ZS, *meeting* SP, SP′ *in* U, U′. *Prove from the definition of a conic that* $SU = SU' = eKI$. *Hence deduce Adams' property, and show that a conic is in perspective with a circle whose centre is at the focus.*

EDITOR.

Draw PM perpendicular to the directrix.

Then $SU : SP :: ZK : ZP$
$:: KI : PM.$
$\therefore SU : KI :: SP : PM,$
$\therefore SU = eKI.$

Similarly $SU' = eKI.$

Adams' property may be easily deduced by considering the limiting position when P′ moves up to and coincides with P, for U′ and U coincide, and KU is then perpendicular to SU.

13. D *is the mid point of the base* BC *of a triangle* ABC. *Any straight line* CFGE *through* C *meets* AD *in* F, AB *in* G, *and the parallel through* A *to* BC *in* E. *If* CH *be drawn perpendicular to* AB, *show that* AB *bisects the angle* EHF.

GREGORY ST. VINCENT.

Obviously AE, AG, AF, AC form a harmonic pencil. Therefore the range E, G, F, C is harmonic. Therefore the pencil HE, HG, HF, HC is harmonic. But the rays HG and HC are at right angles, therefore HG bisects the angle EHF.

14. *Give a simple geometrical demonstration that if the sides of a triangle are at one another as* 4 : 5 : 6, *then the largest angle is double the smallest.*

EDITOR.

Let AB : BC : CA = 4 : 6 : 5.

Bisect angle BAC by AD.

$$\therefore BD : DC :: BA : AC \quad \text{(VI. 3)}$$
$$:: 4 : 5$$
$$\therefore BD : BC :: 4 : 9$$
$$\therefore BD : 6 :: 4 : 9$$
$$\therefore BD : 4 :: 6 : 9$$
$$:: 4 : 6$$
$$\therefore BD : AB :: AB : BC.$$

Therefore the Δ^s ABD and ABC have the angle at B common and the sides about it proportional, and are therefore equiangular.

$$\therefore \text{angle } BAD = C,$$
$$\therefore A = 2C.$$

15. AB *is a given straight line divided into two parts at the point* C. *Show how to find a point* D *in* AC *such that the square on* AD *shall be equal to the rectangle* CD, BD.

GREGORY ST. VINCENT.

Analysis. Bisect BC in O.

$$AD^2 = BD \cdot BC$$
$$= OD^2 - OC^2$$
$$\therefore OC^2 = OD^2 - AD^2.$$

Therefore D lies on a known straight line perpendicular to AB, hence the following construction:—Draw AE perpendicular to AB and equal to OC. Describe circles with centres O, A and radii OE, AO. The common chord of these circles cuts AB at the point required.

16. *Give a simple geometrical demonstration that*

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

and

$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

Describe a circle of unit radius, and draw radii OR, OE, OF, OG at successive angles equal to A (Fig. 5). Draw EH, GL perpendicular to OR. Join GE, cutting OF at M, and MH, cutting OE at N. Draw MK, NT perpendicular to OR, and NP perpendicular to EH. Let MK cut OE at Q.

Then angle ENP = angle NHE = A,

and $MQ = HE = \sin A,$

$PE = NE \sin A = HE \sin^2 A = \sin^3 A,$

$\therefore \sin 3A + \sin A = LG + HE$
$= 2KM$
$= 2\,(KQ + HE)$
$= 4TN$
$= 4HP$
$= 4 \sin A - 4 \sin^3 A,$

$\therefore \sin 3A = 3 \sin A - 4 \sin^3 A.$

Again, similarly,

$OT = \cos^3 A$, and $TH = \cos A - \cos^3 A,$

$\therefore \cos A - \cos 3A = LH = 2KH = 4TH$
$= 4 \cos A - 4 \cos^3 A.$

$\therefore \cos 3A = 4 \cos^3 A - 3 \cos A.$

A somewhat similar solution by R. F. Muirhead was also received.

17. *Solve geometrically the equation* $\cos\theta = \tan\theta$. *Show the connection between the above and the following Diophantine*

problem :—To find three numbers in continued proportion such that the square of the greatest shall be equal to the sum of the squares of the other two. EDITOR.

Take a straight line AB, and divide it at C so that square on AC = rectangle AB, BC (II. 11).

Describe a semicircle on AB, and at C erect a perpendicular to meet it at D. Join AD, DB. Then DAB is the angle required.

For square on BD = rect. AB, BC (VI. 8)
= sq. on AC,
$\therefore \frac{BD}{AD} = \frac{AC}{AD}$.
$\therefore$ tan DAB = cos DAB.

Obviously, since AB . DB = AD^2, therefore AB, AD, DB represent three numbers, which are solutions of the Diophantine problem.

This problem is given by J. de Billy in his *Diophantus Geometra* (1660). It occurs in the trigonometrical form in Hutton's *Miscellanea Mathematica.*

SOLUTIONS OF EXAMINATION QUESTIONS.

(BY A. E. FIELD AND THE EDITOR.)

The Editor will be glad to avail himself of the help of all classes of readers towards making this section of the Gazette *as useful as possible.* MATHEMATICAL TUTORS *are invited to send neat solutions;* STUDENTS *to call attention to classes of problems presenting exceptional difficulties, and* EXAMINERS *who sympathise with us to forward copies of their papers. The help of foreign readers is especially requested in obtaining copies of papers set in the public examinations of other countries.*

28. *Find without direct calculation approximate values of the two roots of the quadratic* $x^2 = 20(x-1)$. [B. A. 1894.]

Since it is obvious that one root is slightly greater than unity, put $x = 1 + y$.

$\therefore (1+y)^2 = 20y$.

But, when y is small, $(1+y)^2 = 1 + 2y$, approximately. Hence, $y = \frac{1}{18}$, and $x = 1\frac{1}{18}$ approximately. Therefore since the sum of the roots is 20, the other root $= 18\frac{17}{18}$ approximately.

29. *In plane trigonometry, if* C = 90°, *sin A = cos B, and tan A = cot B. State and prove the corresponding formulæ for a spherical right-angled triangle.* [B. A. 1894.]

In a spherical right-angled triangle, we have

$$\sin A = \frac{\sin a}{\sin c}, \text{ and } \cos B = \frac{\tan a}{\tan c},$$
$$\therefore \frac{\sin A}{\cos B} = \frac{\cos a}{\cos c} = \frac{1}{\cos b},$$
$$\therefore \sin A \cos b = \cos B.$$

Again, $\tan a = \tan A \sin b$, and $\tan b = \tan B \sin a$,
$\therefore \tan a \tan b = \tan A \tan B \sin a \sin b$,
$\therefore \cot A \cot B = \cos a \cos b$,
$\therefore \cot A \cot B = \cos c$.

These are the corresponding formulæ.

30. *The successive differences between 6-figure logarithms to base 10 of successive integers between 5000 and 5100 are sensibly constant; why is this? Also calculate the difference, assuming* $\log_e 10 = 2{\cdot}302585$. [B. A. 1894.]

$$\log_{10}(x+1) - \log_{10} x = \log_{10}\frac{x+1}{x}$$
$$= \mu \log_e\left(1 + \frac{1}{x}\right), \text{ where } \mu = \frac{1}{\log_e 10}$$
$$= \mu\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \ldots\right)$$
$$= {\cdot}434({\cdot}0002 - {\cdot}00000002 + \ldots) \text{ if } x = 5000$$
$$= {\cdot}0000868, \text{ if } x = 5000,$$

and $= {\cdot}0000851$, if $x = 5100$,

therefore the successive difference for 6-figure logarithms is 87 when $x = 5000$, and 85 when $x = 5100$.

31. *State and illustrate by curves the range of values which* $x^2 + px + q$ *can assume for different values of* x, *distinguishing the several cases.* [B. A. 1894.]

Let $y = x^2 + px + q$; then, writing this, $y = \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$, we see that there are three different cases, according as $q - \frac{p^2}{4}$ is positive, zero, or negative. The graph in each case is a parabola, whose vertex is at the point $\left(-\frac{p}{2}, q - \frac{p^2}{4}\right)$, and whose axis is parallel to the axis of y.

In each case, as x increases from $-\infty$ to $-\frac{p}{2}$, y decreases from ∞ to $q - \frac{p^2}{4}$, its minimum value; and, as x increases through zero from $-\frac{p}{2}$ to ∞, y increases from $q - \frac{p^2}{4}$ to ∞.

In the third case, it should be noticed that $y = 0$, when $2x = -p \pm \sqrt{p^2 - 4q}$.

32. P *is a point on a circle of radius* a, *and* Q *and* R *are two adjacent points on the curve and on the tangent at* P *such that* PR = *arc* PQ = $a\theta$. *If* RQ *produced meet the diameter through* P *in* P′, *obtain an expression for the length of* PP′, *and find its limiting value when* θ *is infinitely small.* [B. A. 1894.]

Let O be the centre of the circle. Join OQ, and from Q drop perpendiculars QM, QN on PP′, PR respectively (Fig. 6). Then angle POQ = θ.

Then QN = MP = OP − OM = $a(1 - \cos\theta)$,
NR = PR − MQ = $a(\theta - \sin\theta)$.

By similar triangles

PP′ : PR :: QN : NR,
$\therefore$ PP′ : $a\theta$ :: 1 − cos θ : θ − sin θ,
$$\therefore \text{PP}' = \frac{a\theta(1-\cos\theta)}{\theta - \sin\theta}.$$

When θ is infinitely small, we have, expanding cos θ and sin θ to the third order of small quantities,

$$1 - \cos\theta = \frac{\theta^2}{2}, \text{ and } \theta - \sin\theta = \frac{\theta^3}{6},$$

therefore, in the limit,

$$\text{PP}' = \frac{a\theta^3}{2} \div \frac{\theta^3}{6} = 3a.$$

33. *There are $2n$ subscribers to a telephone exchange; in how many ways can all the subscribers be in communication, each with another, at the same time?* [B. A. 1894.]

Consider any one subscriber: he can be placed in communication with another in $2n-1$ ways. This having been done, the next subscriber can be placed in communication with any one of the $2n-3$ remaining subscribers. And so on: hence the required number of ways $=(2n-1)(2n-3)(2n-5)\ldots 5.3.1$.

34. *Solve the equations—*

$$x^2+a(2x+y+z)=y^2+b(x+2y+z)=z^2+c(x+y+2z)=(x+y+z)^2.$$

We have $(x+y+z)^2-x^2=a(2x+y+z)$,

$\therefore$ either $2x+y+z=0$ or $y+z=a$, and so on.

Hence (i) $x=\frac{1}{2}(b+c-a)$, $y=\frac{1}{2}(c+a-b)$, $z=\frac{1}{2}(a+b-c)$,

(ii) $x=y=-\frac{1}{2}z=\frac{1}{2}c$, (iii) $y=z=-\frac{1}{2}x=\frac{1}{2}a$,

(iv) $z=x=-\frac{1}{2}y=\frac{1}{2}b$, (v) $x=y=z=0$.

35. *If one root of a rational integral equation be $a+b^{\frac{1}{3}}+c^{\frac{1}{3}}$, show that the equation of least degree satisfying this condition is—*

$$\{(x-a)^3+3b(x-a)-c\}^2-b\{3(x-a)^2+b\}^2=0.$$

We have $x-a-b^{\frac{1}{3}}=c^{\frac{1}{3}}$,

$\therefore$ cubing, $(x-a)^3-3(x-a)^2b^{\frac{1}{3}}+3(x-a)b-b^{\frac{1}{3}}=c$,

$\therefore (x-a)^3+3b(x-a)-c=b^{\frac{1}{3}}\{3(x-a)^2+b\}$,

which, when squared, is the result required.

36. *Show that on the axis of any parabola there is a certain point* K *such that if any chord* PQ *be drawn through it* $1/PK^2+1/QK^2$ *is constant for all positions of the chord.*

Taking the axis of the parabola as initial line and any point on it at a distance k from the vertex as origin, the equation to the curve is

$$r^2\sin^2\theta=4a(k+r\cos\theta)$$

or $$4aku^2+4au\cos\theta-\sin^2\theta=0.$$

Hence $$u_1+u_2=-\frac{\cos\theta}{k},\quad u_1u_2=-\frac{\sin^2\theta}{4ak},$$

$$\therefore u_1^2+u_2^2=\frac{\cos^2\theta}{k^2}+\frac{\sin^2\theta}{2ak},$$

which is constant if $k=2a$.

37. *Solve the equations—*

$$\frac{a^2}{x}+\frac{b^2}{y}=a+b \qquad \text{(i)}.$$

$$(a^2+ab+b^2)(x^2+y^2)-(a^2+b^2)xy=(a^2+b^2)^2 \qquad \text{(ii)}.$$

[Jes. (Oxf.) Schol. 95.]

From (i) $$a^2y+b^2x=(a+b)xy \qquad \text{(iii)}.$$

$$\begin{aligned}\therefore\ &\{(a^2+ab+b^2)(x^2+y^2)-(a^2+b^2)xy\}\{a^2y+b^2x\}^2\\ &=(a^2+b^2)^2(a+b)^2x^2y^2\\ &\quad b^4(a^2+ab+b^2)x^4\\ &+\{2(a^2+ab+b^2)a^2b^2-(a^2+b^2)b^4\}x^3y\\ &-\{2(a^2+b^2)a^2b^2+(a^2+b^2)^2(a+b)^2\}x^2y^2\\ &+\{2(a^2+ab+b^2)a^2b^2-(a^2+b^2)a^4\}xy^3\\ &+a^4(a^2+ab+b^2)y^4=0 \qquad \text{(iv)}.\end{aligned}$$

Now (i) and (ii) are obviously satisfied by $x=a$, $y=b$, and it may easily be verified that they are also satisfied by $x=y=\frac{a^2+b^2}{a+b}$. Hence the left-hand member of (iv) has $bx^2-(a+b)xy+ay^2$ for one of its quadratic factors. Let $lx^2+mxy+ny^2$ be the other. Then

$$l=b^3(a^2+ab+b^2),\quad n=a^3(a^2+ab+b^2),$$

$$bm-l(a+b)=2(a^2+ab+b^2)a^2b^2-(a^2+b^2)b^4,$$

whence $$m=ab(a+b)(2a^2+ab+lb^2).$$

Let k be either of the values of x/y found by solving the quadratic—

$$(a^2+ab+b^2)(b^3x^2+a^3y^2)+ab(a+b)(2a^2+ab+lb^2)xy=0 \qquad \text{(v)},$$

which are both real. Then the remaining solutions are

$$ky=x=\frac{a^2+b^2k}{a+b}.$$

Prof. Steggall obtains the quadratic (iv) by a different artifice. Multiplying $(a+b)xy=(b^2x+a^2y)$ by $(a^2+b^2)(a+b)$, and (ii) by ab, and adding,

$$(a^2+ab+b^2)\{ab(x^2+y^2)+(a^2+b^2)xy\}$$
$$-(a^2+b^2)(a+b)(b^2x+a^2y)-ab(a^2+b^2)^2=0 \qquad \text{(vi)}$$

i.e.

$$\{ax+by-(a^2+b^2)\}\{(a^2+ab+b^2)(bx+ay)+ab(a^2+b^2)\}=0.$$

$ax+by-(a^2+b^2)=0$ leads to the solutions $x=a$, $y=b$, and $x=y=(a^2+b^2)/(a+b)$.

$$(a^2+ab+b^2)(bx+ay)+ab(a^2+b^2)=0,$$

combined with (i), leads to the quadratic (vi). The student's attention is drawn to the *geometrical* connections of the above analytical processes: (i) is a rectangular hyperbola with its asymptotes parallel to the axes of co-ordinates; (ii) is an ellipse whose major axis lies along $x-y=0$, the semi-major axis being $(a^2+b^2)/(a+b)$. The solution $x=y=(a^2+b^2)/(a+b)$ was suggested by a rough graph; (iii) gives the four straight lines joining the origin to the intersections of (i) and (ii), one pair of which we know to be $bx^2-(a+b)xy+ay^2=0$; (vi) was obtained from the consideration that as $ax+by-(a^2+b^2)=0$ was the equation to one of the common chords of (i) and (ii), there must be some value of λ for which the expression—

$$(a^2+ab+b^2)(x^2+y^2)-(a^2+b^2)xy-(a^2+b^2)^2$$
$$+\lambda\{(a+b)xy-(b^2x+a^2y)\}$$

has $ax+by-(a^2+b^2)$ for a factor.

Whence $$\lambda=\frac{(a+b)(a^2+b^2)}{ab}.$$

As $$(a^2+ab+b^2)(bx+ay)+ab(a^2+b^2)=0,$$

and $$ax+by-(a^2+b^2)=0,$$

are equally inclined to the axes, it follows that the four common points of (i) and (ii) are concyclic, as is also obvious from the fact that we can eliminate xy between (ii) and (iii).

QUESTIONS FOR SOLUTION BY STUDENTS.

46. Démontrez que l'équation

$$\sqrt{x-a}+\sqrt{x-b}+\sqrt{x-c}=0,$$

a toujours deux racines réelles et inégales.

PROF. J. NEUBERG.

47. Pour quelles valeurs de m les racines de l'équation

$$x^2-4(1-m)x+(1-m^3)=0$$

sont-elles réelles, inégales ou imaginaires?

PROF. J. NEUBERG.

48. Deux droites AA′ et BB′, non situées dans un même plan, sont coupées par un plan P aux points X et

X′. Lorsque le plan P se meut parallèlement à lui-même quel est le minimum de la droite XX′?

PROF. J. NEUBERG.

49. If $\frac{1}{6841t}$ (scale 11) be converted into an infinite undecimal, prove that the 49,513th figure is 5.

W. P. WORKMAN.

50. Prove that the sum of the digits of every multiple of 2739726 up to the 72nd is 36. EDITOR.

51. If the diameter of a cylinder be d inches, its height h yards, and the s.g. of its material s, show that, approximately, weight in lbs. $= sd^2h(1 + \frac{1}{50})$,

volume in gallons $= \frac{1}{10}d^2h(1 + \frac{1}{50})$.

Hence find weight of solid sphere one foot in diameter whose s.g. = 7·8. PROF. A. LODGE.

52. Four normals are drawn from a given point P to an ellipse meeting it in A, B, C, D. The rectangular hyperbola passing through these five points cuts the directrices of the ellipse in Q and R. Prove that the straight lines PQ, PR pass through the foci of the ellipse.

E. P. ROUSE.

53. A square and a regular hexagon of equal area are taken: in the square is inscribed a circle, in this circle a square, in this square a circle, and so on. In the hexagon is inscribed a circle, in this circle an hexagon, in this hexagon a circle, and so on. Prove that the sum of the areas of the hexagons is double that of the squares.

A. E. FIELD.

54. From the mid point O of the line joining the centres of two given intersecting circles are drawn OK perpendicular to a common tangent, and OL perpendicular to any other straight line passing through the external centre of similitude. If OL meets the common tangent at F, show that $OF : OL :: OK^2 : OL^2$. Hence show that the locus of the mid points of parallel chords to a conic is a straight line. EDITOR.

55. Show that $\frac{a+1}{a-1}$ is an approximate value of $\sqrt{\frac{a+2}{a-2}}$, the approximation being closer as a increases. Hence show that, approximately, $\sqrt{13} = 18/5$; $\sqrt{77} = 351/40$; $\sqrt{2} = 99/70$; $\sqrt{47} = 665/97$. A. M. WILCOX.

56. From a point S on a circle SPM a perpendicular is drawn to the diameter through P, meeting any other chord through P in K. Show that triangles SPK, SPM are similar. Hence show that in a central conic SY, CP intersect on the directrix. EDITOR.

57. Draw graphs illustrating the solution of the equations—

$$x^2 = ax + by,$$
$$y^2 = bx + ay,$$

for different values of a and b. EDITOR.

BOOKS AND MAGAZINES RECEIVED.

Geometrical Conics. Part I. *The Parabola.* Part II. *The Central Conic.* By the Rev. J. J. MILNE, M.A., and R. F. DAVIS, M.A. (Macmillan & Co.) Part I. was published in 1890, and has already received favourable notices from the educational press. The complete work now appears in one volume. In Part II. the authors have, as the title implies, treated the ellipse and hyperbola together. In the proofs the authors claim to have confined themselves "entirely to the methods of ancient geometry, the via Euclideana et Apolloniana," and to "have made no use of the so-called auxiliary circle of the ellipse, or the conjugate hyperbola, or in fact of any auxiliary curve in proving the properties common to both species."

The book is a distinct addition to the library of the mathematical student. For while opinions may differ as to the extent to which a student who has "mastered his Euclid" should in a course of Geometrical Conics be introduced to the methods of Modern Geometry, there is a distinct advantage in being able to consult a treatise in which such advanced theorems as *ad quatuor lineas* and the famous discovery of Apollonius on the feet of four concurrent normals, are reached by neat Euclidean demonstrations. We are inclined to value a course of Geometrical Conics quite as much for its affording a means of introducing the student to *new ideas* (such as those of Limits, Curvature, Projection, Transformation, etc.) as for its giving him *new facts*, and would point out that there is less need for "Euclidean grinds" now that modern methods are allowed and even encouraged in the Euclid and Conics paper of the Mathematical Tripos. Granted, however, the ideal of the authors, it must be admitted that they have gone the right way to attain it, and that they have shown what a powerful weapon the old geometry is in skilful hands. The treatment consists in a great measure of a return to that of Apollonius, to whom constant references are made. In consequence, comparatively little use is made of the directrix property, so little in fact that the directrix only appears in 19 out of about 120 diagrams with which the work is illustrated, 11 of these 19 belonging to the parabola. One excellent feature is the juxtaposition of the three possible definitions of a central conic ($S'P \pm SP = 2a$, $SP = ePM$, $PN^2 \propto AN \cdot A'N$), and the proofs that if any one of these be taken as a basis the other two can be deduced as necessary properties. Separate sections are given to the consideration of Diameters, Asymptotes, Normals, Confocal Conics, and valuable addenda introduce the student to the connection between Conic Sections and some of the historic problems of plane geometry, such as the Trisection of a circular arc, the finding of two mean proportionals, Alhazen's, and to some later theorems, such as Pascal's, Maclaurin's, and Stewart's. The treatment of conics as envelopes is very carefully done. We are glad to see Chasles' neat construction* for finding the axes of an ellipse when a pair of conjugate diameters are known, given in full, as it deserves to be, with an excellent figure. We suggest that in a new edition the construction depending on the description of the ellipse by a "trammel" should

* The subject of a historical note by Prof. Genese in our last number.

Fig 1.

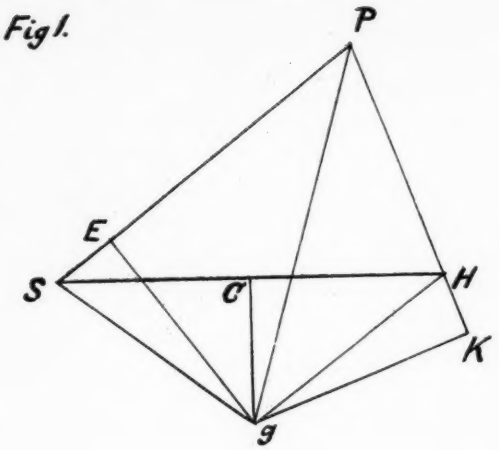
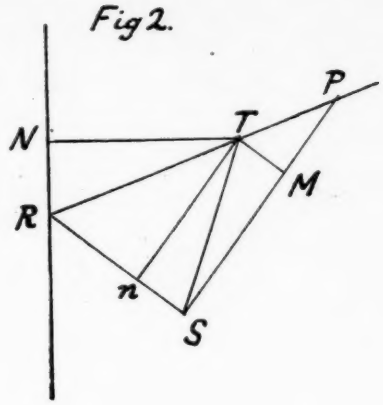


Fig 2.



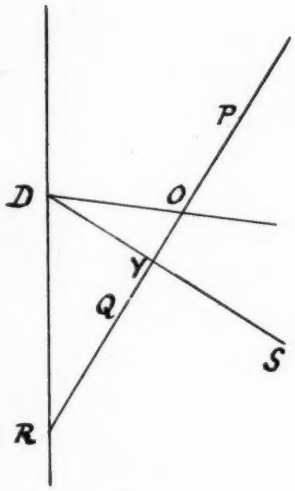


Fig 3.

Fig 4.

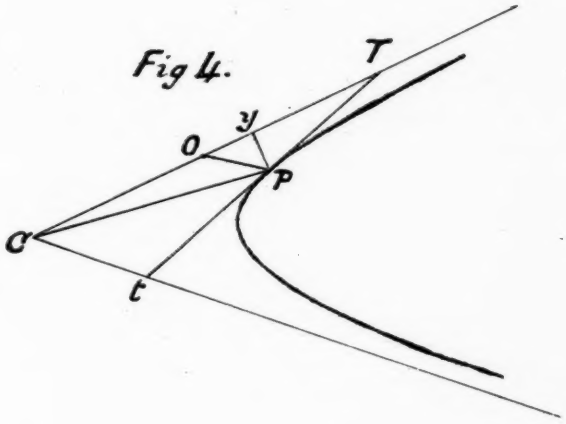


Fig 5.

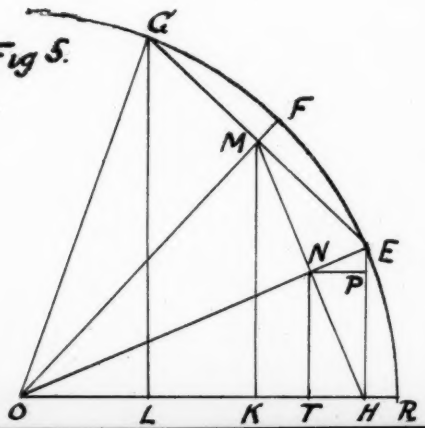
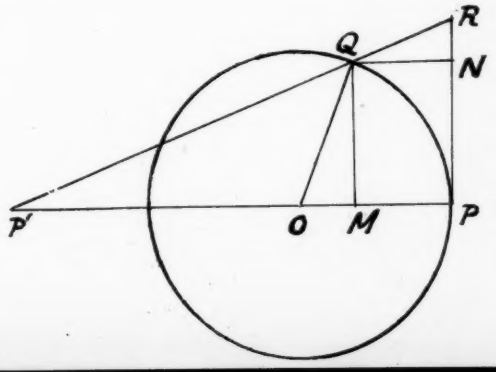


Fig 6.



be given as an alternative, and that the close connection between the two should be pointed out. Among the miscellaneous theorems at the end occurs Mr. G. T. Bennett's proof of the existence of the Frégier point,* and the determination of the constant k in the trilinear equation $\alpha\beta = k\gamma\delta$.†

The work does not appear to contain any section on the General Conic, and we have not found any solution of the problem which Boscovich took as his fundamental one, To find where a given straight line cuts a *Conic whose focus, directrix, and eccentricity are given.*‡ This seems quite as worthy of a niche to itself as Stewart's or Maclaurin's Theorem.

Geometrical Conics. By C. SMITH. (Macmillan & Co.). Those who have studied the *Algebra* and the *Conic Sections* (Analytical) by the same author will turn to his new work in the well-grounded expectation of finding a clear and well-arranged treatise, and they will not be disappointed. The arrangement and selection of fundamental propositions is judicious, the proofs neat, the diagrams as a rule well executed, and the type excellent, bold Clarendons calling attention to important phrases and propositions. The Parabola, Ellipse, and Hyperbola have entirely separate sections devoted to them, rightly, in our opinion, in a treatise intended chiefly for beginners. These sections are preceded by one on the General Conic, which, however, need not be taken first if a student prefers to begin with the Parabola. We are glad to see that Mr. Smith, like Messrs. Milne and Davis, has "discarded the usual method of treating a hyperbola as if it were two conics, the curve itself and the conjugate hyperbola." Dr. Taylor has long advocated this reform,§ and it is satisfactory to find the excrescence gradually disappearing from text-books. It is to be hoped that examiners will accelerate the process by ceasing to set questions on the conjugate hyperbola. Probably they will not be sorry to be relieved from the task of looking through reproductions by hurried students of diagrams and demonstrations which are repulsive enough even when correctly executed.

Mr. Smith, in the chapter on the General Conics, follows the method adopted by Leslie and Dr. Taylor (after Boscovich) of using the Eccentric Circle (see Nos. 1 and 3 of the *Mathematical Gazette*) for investigating the ratios of rectangles contained by the segments of chords, but has recourse to the Adams' circle for drawing tangents from an external point, and makes no use of either of them in solving the fundamental problem of Boscovich. It would have given more unity and coherence to the chapter to have used either Adams' or Boscovich's circle throughout, while the connection between the two circles would have afforded good material for exercises, and simple illustrations of some of the methods of Modern Geometry.

The analogies between the properties of the polar *of*, and the tangent *at* a point are well presented, and some excellent addenda draw attention to the properties of Envelopes, Coaxial Parabolas, Confocal Conics, etc. We note two useful properties not often given in works on Conics, viz. gE perpendicular to SP in the Ellipse, p. 108, and RY = SP in the Hyperbola, p. 151. Dr. Taylor was, we believe, the first to point out and utilise these. (See his *Elementary Geometry of Conics* and his *New Treatment of the Hyperbola* in A.I.G.T. Report.) They well deserve to be better known and more used. The diagram for the former is exceptionally faulty. The construction given for finding the axes of an ellipse when a pair of conjugate diameters is given depends on the values of PG and Pg, by which the mid point O of Gg is found; but this is not so neat as the one connected with the "trammel," which we have referred to elsewhere. The least satisfactory portion of the book is that on Anharmonics and Conical Projection. There seems to be no vital connection between the elementary and the advanced sections.

We are of opinion that the space devoted to Conical Projection might have been more usefully given to Plane Perspective and other plane transformations, the previous sections on Orthogonal Projection, Adams' Circle, Similar Conics and the Eccentric Circles being used for illustrations of the general methods described.

Dynamics—Mechanics. By R. T. GLAZEBROOK, M.A., Cambridge University Press. These form part of a series of *Cambridge Natural Science Manuals*, written from the point of view of one who believes *that the most satisfactory method of teaching the natural sciences is by experiments which can be performed by the learners themselves*, a principle which has our emphatic approval. There is a welcome freshness about the illustrations, which no doubt arises from the fact that the illustrations have been mostly drawn from apparatus used by classes in the Cavendish Laboratory. We join with the author in the hope he expresses that the systematic experimental work described may, through the means of these excellent manuals, be extended in colleges and schools. Some further remarks in the preface are so apt that they deserve quotation *in extenso*: "When questions dealing with momentum, force, and energy come to be considered, two courses at least are open to the teacher. It is possible to make the whole subject purely deductive; we may start with some definitions and axioms—laws of motion, either as Newton gave them, or in some modern dress—and from these laws we may deduce the behaviour of bodies under various circumstances.

"Another and more instructive method, it seems to me, is to attempt to follow the track of the founders of mechanics, to examine the circumstances of the motion of bodies in certain simple cases, in the endeavour to discover the laws to which they are subject. . . . Mechanics is too often taught as a branch of pure mathematics. If the student can be led up to see, in its fundamental principles, a development of the consequences of measurements he has made himself, his interest in his work is at once aroused; he is taught to think about the *physical meaning*

* See 16th General Report of the A.I.G.T.
† See *Mathematical Gazette*, pp. 21, 33.
‡ See *Mathematical Gazette*, Nos. 1 and 3.
§ See Report of the A.I.G.T. for 1884.

of the various steps he takes, and *not merely to employ certain rules and formulæ* in order to solve a problem." We endorse the view taken by the author, and are of opinion that it cannot be too strongly impressed on teachers. At the same time there is no doubt that the course of experimental study recommended may be very advantageously *followed* by a purely deductive one, the mind of the student having been prepared by his previous training to form adequate mental concepts of the definitions and axioms on which the fabric of theory is to be founded. It must not be supposed, however, that the author confines himself merely to descriptions of experiments. The laws and theorems are clearly stated and well illustrated by analytical statements and numerical examples.

An Elementary Text-Book on Mechanics. Bk. I. *Dynamics.* Bk. II. *Statics.* By W. BRIGGS, M.A., and G. H. BRYAN, M.A. (University Tutorial Series.) Though not written quite on the same lines as Mr. Glazebrook's, this is an excellent treatise. It is marked by the same clearness of exposition and careful attention to the needs of a beginner which characterise the well-known *Co-ordinate Geometry* by the same authors. It will be found especially useful by students who have to master these subjects with little or no aid from tutors.

Integral Calculus for Beginners. By JOSEPH EDWARDS, M.A. This forms an excellent companion volume to the *Differential Calculus for Beginners* by the same author. It is a clearly written systematic treatise, and can be heartily recommended. The analogies of the *hyperbolic* (or as Mr. Hayward calls them, the *ex-circular*) functions, with the more common trigonometrical functions, are pointed out and used to make some of the forms for integration more easily remembered. Geometrical illustrations are given of the various processes: *e.g.*, of that of *Integration by Parts*, on pp. 47, 48. The value of the work is enhanced by the addition of a section on *Differential Equations*, sufficient for the purposes of those students who wish to read *Analytical Statics, Dynamics of a Particle*, and *Elementary Rigid.*

Arithmetic for Schools (new edition), by the Rev. J. B. Lock,—a much improved edition of an already serviceable text-book. The innovations will be familiar enough to members of the A.I.G.T., who will be pleased to see long-advocated reforms in arithmetic gradually urged on teachers by such a well-known writer as Mr. Lock. Among these we notice especially complementary addition (in other words "*shop*" *subtraction*), the actuaries' rule for decimalising money, Horner's method for evolution. In all these matters Mr. Lock goes as far in the direction of reform as he can well be expected to do. He speaks with no uncertain sound on the claims of the "Italian" method of division to universal adoption, and advocates, rather too mildly in our opinion, the use of the digits of higher order first in multiplication. Periods of transition are always uncomfortable, but it is time now that the difficulty should be met in all schools of repute, and an end put to the inculcation *ab initio* of methods of work which any one who wishes to carry on serious computation must discard. Experience has shown that the difficulties attending the change advocated are by no means insurmountable.

Euclid's Elements of Geometry. Bks. I.-IV. By P. GHOSH. 16th edition, revised and enlarged by A. S. GHOSH, F.R.A.S. (Patrick Press, Calcutta.) An excellent edition, containing numerous exercises, with useful hints for solution, and addenda introducing the student to modern results. Among these is a valuable section on maxima and minima, in which the method of the "Coincidence of Equal Values" is well expounded.

Arithmetic for Schools and Colleges. By P. GHOSH. 19th edition, revised and enlarged by A. S. GHOSH. (Patrick Press, Calcutta.) A serviceable treatise, based on sound principles. It is well furnished with examples, and contains more than 100 pages of Indian University Examination papers. The chapter on Coinage Systems and Exchange seems especially good. We should like, in a future edition, to see the work made still more valuable by alteration in the directions indicated in our notice of Mr. Lock's arithmetic, and treated with some detail in the *Notes on Arithmetic*, published in the Nineteenth A.I.G.T. Report.

Pedal and Antipedal Triangles. By A. S. GHOSH. (Patrick Press, Calcutta.) An interesting little paper. The author has been anticipated on some points, but his mode of presentation is fresh, and the properties in the latter part are new.

Phases of Perigal's Retrogressive Kinematic Parabola. (Bowles and Son, Castle Street, Finsbury.) A collection of six beautifully executed diagrams of the curve $y = a\cos\phi$, $x = a\cos 2\phi$, dedicated by the venerable author to the A.I.G.T. The title-page informs us that "The *kinematic* curve, of which the *retrogressive* parabola is a limit, was discovered by Mr. Perigal in 1835, and produced from continuous motion by him in 1840."

Geometry for Grammar Schools. By G. HUNT. (D. G. Heath and Co., Boston.) A book much on the same lines as Spencer's *Inventional Geometry*, and, like it, well adapted, in the hands of an able teacher, to lead on from the methods of the Kindergarten. We hope the time is not far distant when the use of some such course as this will be considered an absolutely necessary introductory training for junior classes before they are confronted with a systematic course of deductive geometry. The English market is supplied by Isbister and Co.

Le Scienze Esatte nell' Antica Grecia (Gino Loria). Lib. II. The book is devoted to "the golden period of Greek Geometry."

The American Mathematical Monthly. Jan., March, April 1895.

The Mathematical Magazine. Jan. 1895.

Journal de Mathematiques Elementaires. March–May 1895.

Periodico di Matematica. Jan.-Feb., Mar.-April 1895.

U. S. Report of the Commissioner of Education. 1891-92.